# Compositional Bernoulli numbers

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#### Abstract

We define and study the combinatorial properties of compositional Bernoulli numbers and polynomials within the framework of rational combinatorics.

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### 1 Introduction

Hitherto combinatorial analysis has focus on the study of natural numbers, yet the time has come to address other numerical structures by combinatorial means. In [4] we proposed a framework for the study of the combinatorics of rational numbers which we review below; first we illustrate how it works with a simple example. Looking at sequences  $1, 2, 4, 8, \dots, 2^n, \dots$ , and  $1, 1, 2, 6, 24, \dots, n!, \dots$ , combinatorialists will agree that they count the number of subsets and the number of permutations of a set with n elements, respectively. It is however less clear what a sequence such as

$$1, 2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}, \cdots, \frac{2^n}{n!}, \cdots$$

might count. The main difference is that whereas the former are sequences of natural numbers, the later is a sequence of rational numbers. One can assign combinatorial meaning to sequences of rational numbers using the notion of cardinality of groupoids introduced by Baez and Dolan in [1]. In order to find out what the sequence above "counts" one should find a sequence of groupoids  $x_0, x_1, \dots, x_n, \dots$  such that  $|x_n| = \frac{2^n}{n!}$ . For  $n \ge 0$ , let  $x_n$  be the groupoid whose set of objects  $Ob(x_n)$  is the collection of subsets of  $[n] = \{1, 2, ..., n\}$ . Morphisms in  $x_n$  from a to b are bijections  $\alpha : [n] \longrightarrow [n]$  such that  $\alpha(a) = b$ . By definition the cardinality of  $x_n$  is given by

$$|x_n| = \sum_{a \in D(x_n)} \frac{1}{|x_n(a,a)|},$$

where  $D(x_n)$  is the set of isomorphisms classes of objects in  $x_n$ , and  $x_n(a, a)$  is the set of morphisms from a to a in  $x_n$ . In the present case we have that

$$|x_n| = \sum_{k=0}^n \frac{1}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} = \frac{2^n}{n!},$$

and thus the sequence of groupoids  $x_n$  provides a combinatorial interpretation for the sequence of rational numbers  $\frac{2^n}{n!}$ . The goal of rational combinatorics, from this point of view, is to uncover the relationship between sequences of rational numbers and sequences of finite groupoids.

We shall actually adopt the functorial viewpoint of Joyal [13], and work with rational species instead of sequences of groupoids. For a comprehensive study of combinatorial species the reader may consult [2]. A presentation of the theory of species in a categorical context [14] with applications to non-commutative spaces is given in [10]. Rational species were introduced in [4] and we shall adopt the notation and conventions of that paper. Further applications of the theory of rational species are developed in [5] and in the forthcoming works [6, 11]. The main ingredients in the definition of the category of rational species are  $\mathbb B$  the category of finite sets and bijections, and gpd the category of finite groupoids. An object of gpd is a category G such that all its morphisms are invertible, Ob(G) is a finite set, and for  $x, y \in Ob(G)$  the set G(x,y) of morphisms from x to y is also finite. Disjoint union  $\square$  and Cartesian product  $\times$  give gdp a couple of monoidal structures with units  $\emptyset$ , empty groupoid, and 1, groupoid with one object and one morphism, respectively. Cardinality for groupoids yields a valuation map  $|\cdot|: Ob(gpd) \longrightarrow \mathbb{Q}_+$  with values in the semi-ring of non-negative rational numbers which satisfies: |x| = |y| if x and y are isomorphic,  $|x \square y| = |x| + |y|$ ,  $|x \times y| = |x||y|$ ,  $|\emptyset| = 0$ , and |1| = 1.

The category of non-negative rational species  $gpd^{\mathbb{B}}$  is the category of functors from  $\mathbb{B}$  to gpd; morphisms in  $gpd^{\mathbb{B}}$  are natural transformations. Monoidal structures sum and product on  $gpd^{\mathbb{B}}$  are given by  $(F+G)(x)=F(x)\sqcup G(x)$  and

$$(FG)(x) = \bigsqcup_{x_1 \sqcup x_2 = x} F(x_1) \times G(x_2),$$

where F and G are rational species, and  $x, x_1, x_2$  are finite sets. Units for sum and product are 0 the species sending a finite set into  $\emptyset$ , and 1 the species sending a non-empty set into  $\emptyset$  and the empty set into 1. The valuation map  $|\cdot|: Ob(gpd^{\mathbb{B}}) \longrightarrow \mathbb{Q}_+[[x]]$  given by  $|F| = \sum_{n=0}^{\infty} F([n]) \frac{x^n}{n!}$ , where  $\mathbb{Q}_+[[x]]$  is the semi-ring of formal power series with coefficients in  $\mathbb{Q}_+$ , is such that |F| = |G| if F and G are isomorphic, |F + G| = |F| + |G|, |FG| = |F||G|, |1| = 1, and |0| = 0. The valuation |F| of a species F is called its generating series. Thus the main problem of rational combinatorics is: given a non-negative rational species F find its generating series  $|F| \in \mathbb{Q}_+[[x]]$ . We also consider the inverse problem: given  $f \in \mathbb{Q}_+[[x]]$ , find a nice rational species F such that |F| = f. For example, consider the hyper-exponential power series  $e_k \in \mathbb{Q}_+[[x]]$  given by

$$e_k = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^k}.$$

Clearly  $e_1$  is the exponential series and, for  $k \geq 2$ ,  $e_k$  is a divided power series with rational coefficients. Let  $E_k : \mathbb{B} \longrightarrow gpd$  be such that for a finite setx, objects in  $E_k(x)$  are tuples

 $(a_1, \dots, a_{k-1}) \in P(x)^{k-1}$  where P(x) is the set of subsets of x. Morphisms in  $E_k(x)$  from  $(a_1, \dots, a_{k-1})$  to  $(b_1, \dots, b_{k-1})$  are tuples  $(\alpha_1, \dots, \alpha_{k-1})$  where  $\alpha_i$  is a permutation of x such that  $\alpha_i(a_i) = b_i$ . The generating series of  $E_k$  is given by

$$|E_k| = \sum_{n=0}^{\infty} |E_k([n])| \frac{x^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{s_1, \dots, s_{k-1}=1}^n \frac{1}{\prod_i s_i! (n-s_i)!} \right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^{(k-1)n}}{(n!)^{k-1}} \frac{x^n}{n!} = e_k(2^{k-1}x).$$

Thus  $E_k$  provides a combinatorial interpretation for the formal power series  $e_k(2^{k-1}x)$ .

The category of non-negative rational species let us define combinatorial interpretations for sequences of non-negative rational numbers. However, both Bernoulli and compositional Bernoulli numbers require that we are able to consider the combinatorics of sequences of arbitrary rational numbers, including negative ones. Let  $\mathbb{Z}_2$ -gpd be the category of  $\mathbb{Z}_2$ -graded finite groupoids, i.e. finite groupoids G together with a map  $Ob(G) \longrightarrow \mathbb{Z}_2$  sending  $x \in Ob(G)$  to  $\overline{x} \in \mathbb{Z}_2$ , such that if there is a morphisms in G from x to y then  $\overline{x} = \overline{y}$ . Morphisms in  $\mathbb{Z}_2$ -gpd are grading preserving morphisms in gpd.  $\mathbb{Z}_2$ -gpd has monoidal structures disjoint union and Cartesian product, where the grading on the disjoint union of groupoids is the disjoint union of the respective gradings, and the grading on the Cartesian product of groupoids G and G is G in G is G and G is G is G and G is G is G is G in G is G in G is G in G in G in G in G is G in G in

$$|G| = \sum_{x \in D(G)} \frac{(-1)^{\overline{x}}}{|G(x,x)|}$$

is such that for all  $\mathbb{Z}_2$ -graded groupoids G and H we have that: |G| = |H| if G and H are isomorphic,  $|G \sqcup H| = |G| + |H|$ ,  $|G \times H| = |G||H|$ ,  $|\emptyset| = 0$ , and |1| = 1. There is a negative functor  $-: \mathbb{Z}_2$ - $gpd^{\mathbb{B}} \longrightarrow \mathbb{Z}_2$ - $gpd^{\mathbb{B}}$  which is the identity both on objects and morphisms but  $\overline{-x} = \overline{x} + 1$ . The category of rational species  $\mathbb{Z}_2$ - $gpd^{\mathbb{B}}$  is the category of functors from  $\mathbb{B}$  to  $\mathbb{Z}_2$ -gpd. One defines monoidal structures sum and product on  $\mathbb{Z}_2$ - $gpd^{\mathbb{B}}$  and the valuation map  $|\cdot|: \mathbb{Z}_2$ - $gpd^{\mathbb{B}} \longrightarrow \mathbb{Q}[[x]]$  in complete analogy with the case of non-negative rational species; the resulting structures enjoy similar properties to those stated for non-negative rational species.

The rest of this work is organized as follows. In Section 2 we provide a combinatorial interpretation for Gauss hypergeometric functions with rational parameters. In Sections 3 and 4 we introduce a generalization of Bernoulli numbers and provide combinatorial interpretation for such numbers; we specialize our construction to a variety of interesting examples. In Section 5 we study the combinatorics of Bernoulli polynomials. Section 6 contains the main results of this work, namely, we introduce compositional Bernoulli numbers and provide combinatorial interpretation for such numbers. In Section 7 we discuss compositional Bernoulli polynomials.

# 2 On the combinatorics of Gauss hypergeometric functions

In this section we study the combinatorics of Gauss hypergeometric functions with rational parameters generalizing the results of [3] where the case of positive rational parameters was tackle. Consider formal power series of the form

$$h(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f}) = \sum_{n=0}^{\infty} \frac{(\pm \frac{a}{b})_n (\pm \frac{c}{d})_n}{(\pm \frac{e}{f})_n} \frac{x^n}{n!},$$

where a, b, c, d, e, f are positive natural numbers,  $\pm$  indicates that a choice of sign has been made, and  $(x)_n$  is the Pochhammer symbol  $(x)_n = x(x+1)(x+2)...(x+n-1)$  also known as the increasing factorial [15]. We also need the Pochhammer k-symbol

$$(x)_{n,k} = x(x+k)(x+2k)...(x+(n-1)k)$$

introduced in [11] and further applied in [8, 9, 12]. We proceed to construct functors

$$H(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f}) : \mathbb{B} \longrightarrow \mathbb{Z}_{2}\text{-}gpd,$$

such that  $|H(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f})| = h(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f})$ . The functor  $H(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f})$  is given on a finite set x by

$$H(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f})(x) = (\pm [a])_{|x|,[b]}(\pm [c])_{|x|,[d]} \overline{\mathbb{Z}}_b^x \overline{\mathbb{Z}}_d^x \overline{\mathbb{Z}}_{\pm e,|x|,f}[f]^x,$$

where we are making use the following conventions:

- 1. A set x may be regarded as the groupoid whose set of objects is x and whose morphisms are identities.
- 2. If G is a group then  $\overline{G}$  is the groupoid with a unique object 1 and  $\overline{G}(1,1) = G$ .
- 3. If G is a groupoid and x a finite set then  $G^x$  is the groupoid whose objects are maps from x into Ob(G), morphisms in  $G^x$  from f to g are given by  $G^x(f,g) = \prod_{a \in Ob(G)} G(f(a),g(a))$ . It is easy to see that  $|G^x| = |G|^{|x|}$ .
- 4. If G and K are groupoids then  $(G)_{n,K} = \prod_{i=0}^{n-1} (G \sqcup (K \times [i]))$ . One can show that  $|(G)_{n,K}| = (|G|)_{n,|K|}$ .
- 5. For  $n \geq 1$ ,  $\mathbb{Z}_n$  denotes the cyclic group with n elements; we also set  $\overline{\mathbb{Z}}_{-n} = -\overline{\mathbb{Z}}_n$ . For  $m \in \mathbb{Z} \setminus \{0\}$  and  $n, l \in \mathbb{N}$  we set  $\overline{\mathbb{Z}}_{m,n,l} = \prod_{i=0}^{n-1} \overline{\mathbb{Z}}_{m+il}$ . One can show that  $|\overline{\mathbb{Z}}_{m,n,l}| = \frac{1}{(m)_{n,l}}$ .

#### Theorem 1.

$$|H(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f})| = h(\pm \frac{a}{b}, \pm \frac{c}{d}; \pm \frac{e}{f}).$$

Proof.

$$|(\pm[a])_{n,[b]}(\pm[c])_{n,[d]}\overline{\mathbb{Z}}_{b}^{[n]}\overline{\mathbb{Z}}_{d}^{[n]}\overline{\mathbb{Z}}_{\pm e,n,f}[f]^{[n]}| = \frac{(\pm a)_{n,b}}{b^{n}}\frac{(\pm c)_{n,d}}{d^{n}}\frac{f^{n}}{(\pm e)_{n,f}} = \frac{(\pm \frac{a}{b})_{n}(\pm \frac{c}{d})_{n}}{(\pm \frac{e}{t})_{n}}.$$

### 3 Combinatorics of Bernoulli numbers

We introduce a generalization of Bernoulli numbers which may be motivated as follows. Suppose we are interested in finding a right inverse for a linear operator  $O: V \longrightarrow V$ , i.e. an operator  $G: V \longrightarrow V$  such that O(G(v)) = v for  $v \in V$ . Assume O can be written as

$$O = f(D) - \pi_N(f)(D),$$

where  $D:V\longrightarrow V$  is a linear map for which a right inverse  $I:V\longrightarrow V$  is known,  $f\in\mathbb{Q}[[x]]$  is a formal power series such that  $f_N=1$ , and for  $N\geq 1$ , and the N-projection map

$$\pi_N: \mathbb{Q}[[x]] \to \mathbb{Q}[[x]]/(x^N)$$

is given by

$$\pi_N\left(\sum_{n=0}^{\infty} f_n \frac{x^n}{n!}\right) = \sum_{n=0}^{N-1} f_n \frac{x^n}{n!}.$$

**Definition 2.** For  $f = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$  and  $N \geq 0$  such that  $f_N = 1$ , the Bernoulli numbers  $B_{N,n}^f$  associated with f and N are given by

$$\sum_{n=0}^{\infty} B_{N,n}^f \frac{x^n}{n!} = \frac{x^N/N!}{f - \pi_N f} = \left( N! \sum_{n=0}^{\infty} \frac{n! f_{n+N}}{(n+N)!} \frac{x^n}{n!} \right)^{-1}.$$

**Theorem 3.** Under the above conditions a right inverse G for O is given by

$$G = N! \sum_{n=0}^{\infty} B_{N,n}^f \frac{D^n \circ I^N}{n!}.$$

Classical Bernoulli numbers  $B_n = B_{1,n}^{e^x}$  arise when computing a right inverse S for the finite difference operator  $\Delta(f) = f(x+1) - f(x)$ , which acts on functions depending on a real variable x. Let  $D = \frac{\partial}{\partial x}$  and R be a right inverse for D, i.e. R is the Riemann integral. Since  $\Delta = e^D - 1$ , then S is given by

$$S = R + \sum_{n=1}^{\infty} B_n \frac{D^{n-1}}{n!} \quad \text{where} \quad \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

By definition Bernoulli numbers  $B_{N,n}^f$  satisfy the identities

$$\sum_{k=0}^{n-N} \binom{n}{k} f_{n-k} B_{N,k}^f = \delta_{n,N} \text{ for } n \ge N,$$

and thus can be computed using the recursions:

$$B_{N,n}^f = \binom{N+n}{n}^{-1} \sum_{k=0}^{n-1} \binom{N+n}{k} f_{N+n-k} B_{N,k}^f.$$

We provide a combinatorial interpretation for Bernoulli numbers  $B_{N,n}^f$  assuming that a combinatorial interpretation for f is known, that is, given  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd we construct  $B_N^F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd such that

$$|B_N^F| = \frac{x^N/N!}{|F| - \pi_N|F|}.$$

Consider first the problem of finding a combinatorial interpretation for  $f^{-1}$  assuming that a combinatorial interpretation for f is known. For  $F \in \mathbb{Z}_2$ - $gpd^{\mathbb{B}}$  such that  $F(\emptyset) = 0$ , let  $(1+F)^{-1}: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd send  $x \in \mathbb{B}$  into

$$(1+F)^{-1}(x) = \bigsqcup_{x_1 \sqcup \dots \sqcup x_k = x, k > 1} (-1)^k \prod_{i=1}^k F(x_i).$$

**Proposition 4.**  $(1+F)^{-1}: \mathbb{B}^n \longrightarrow \mathbb{Z}_2\text{-}gpd \text{ satisfies } |(1+F)^{-1}| = (1+|F|)^{-1}.$ 

The decreasing factorial rational species  $\mathbb{Z}_{(N)}: \mathbb{B} \longrightarrow gpd$  is such that for  $x \in \mathbb{B}$  we have  $Ob(\mathbb{Z}_{(N)}(x)) = \{x\}$  if  $|x| \geq N$  and empty otherwise. For  $|x| \geq N$  morphisms in  $\mathbb{Z}_{(N)}(x)$  are given by

$$\mathbb{Z}_{(N)}(x,x) = \mathbb{Z}_{|x|} \times \mathbb{Z}_{|x|-1} \times \cdots \times \mathbb{Z}_{|x|-N+1}.$$

The derivative  $\partial F$  of a species F is given by  $\partial F(x) = F(x \sqcup \{x\})$  for  $x \in \mathbb{B}$ ; it is easy to check that  $|\partial F| = \partial |F|$ . For  $n \geq 1$  and G a groupoid, let nG be the groupoid  $G \sqcup ... \sqcup G$  (n copies).

**Proposition 5.** Let  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd be a rational species such that F([N]) = 1, then

$$|1+N!\partial^N (F\times \mathbb{Z}_{(N)})|=N!\frac{|F|-\pi_N|F|}{x^N}.$$

*Proof.* For n = 0 the desired result is obvious. For  $n \ge 1$  we have

$$|1 + N! \partial^{N} (F \times \mathbb{Z}_{(N)}) ([n])| = N! |F([n] \sqcup [N])| |\mathbb{Z}_{(N)} ([n] \sqcup [N])| = N! |F|_{n+N} \frac{n!}{(n+N)!},$$

which is the *n*-th coefficient of the divided power series  $N! \frac{|F| - \pi_N(|F|)}{x^N}$ .

**Theorem 6.** Let  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd be such that F([n]) = 1. The species  $B_N^F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd sending  $x \in \mathbb{B}$  into

$$B_N^F(x) = \bigsqcup_{x_1 \sqcup \cdots \sqcup x_k = x, k \ge 1} (-N!)^k \prod_{i=1}^k F \times \mathbb{Z}_{(N)}(x_i \sqcup [N]),$$

is such that  $\left|B_N^F\right| = \sum_{n=0}^{\infty} B_{N,n}^{|F|} \frac{x^n}{n!}$ .

*Proof.* Follows from Proposition 4 and the identity  $|B_N^F| = |1 + N! \partial^N (F \times \mathbb{Z}_{(N)})|^{-1}$ .

Next we compare  $\Delta$  with  $D = \frac{\partial}{\partial x}$ ; for example we may like to know a right inverse for the operator  $\Delta - D = e^D - 1 - D$ , or more generally a right inverse for the operator  $e^D - \pi_N(e^D)$ . According to Theorem 3 a right inverse for  $e^D - \pi_N(e^D)$  is given by

$$G = N! \sum_{n=1}^{N-1} B_{N,n} \frac{I^{N-n}}{n!} + N! \sum_{n=N}^{\infty} B_{N,n} \frac{D^{n-N}}{n!},$$

where the Bernoulli numbers  $B_{N,n}$  are such that

$$\frac{x^{N}/N!}{e^{x}-1-x} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^{n}}{n!}.$$

Explicitly the first Bernoulli numbers  $B_{N,n} = B_n$  are shown in the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$B_n$	1	-1/2	1/6	0	-1/30	0	1/42	0	-1/30	0	5/66	0	-691/2730	0	7/6

The first Bernoulli numbers  $B_{2,n}$  are given by:

n	0	1	2	3	4	5	6	7	8	9
$B_{2,n}$	1	-1/3	1/18	1/90	-1/270	-5/1134	-1/5670	7/2430	13/7290	-307/133650

The exponential species E given by  $E(x) = \{x\}$  for  $x \in \mathbb{B}$  is such that  $|E| = e^x$ . Theorem 6 implies that the generating series of the species  $B_N : \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd sending  $x \in \mathbb{B}$  into

$$B_N(x) = \bigsqcup_{x_1 \sqcup ... \sqcup x_k = x, k \ge 1} (-N!)^k \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [N])$$

is  $\sum_{n=0}^{\infty} B_{N,n}^f \frac{x^n}{n!}$ . Thus we have obtained a combinatorial interpretation for  $B_{N,n}$  in terms of the cardinality of  $\mathbb{Z}_2$ -graded groupoids:

$$B_{N,n} = |B_N([n])| = n! \sum_{a_1 + \dots + a_k = n, k \ge 1} \frac{(-N!)^k}{(a_1 + N)! \dots (a_k + N)!}.$$

Let us consider Bernoulli numbers associated with the sine function and N=2L+1 an odd number. For N=1 we obtain Bernoulli numbers  $B_{1,n}^{sin}$  given by

$$\frac{x}{\sin(x)} = \sum_{n=0}^{\infty} B_{1,n}^{\sin} \frac{x^n}{n!}.$$

The first few values of the sequence  $B_{1,n}^{sin}$  are shown in the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B_{1,n}^{\sin}$	1	0	1/3	0	7/15	0	31/21	0	127/15	0	2555/33	0	1414477/1365	0

One can show that  $B_{1,2n}^{sin} = (-1)^{n-1} (2^{2n} - 2) B_{2n}$  and  $B_{1,2n+1}^{sin} = 0$ . For N = 3 we obtain the Bernoulli numbers  $B_{2,n}^{sin}$  given by

$$\frac{x^3/3!}{\sin(x) - x} = \sum_{n=0}^{\infty} B_{3,n}^{\sin} \frac{x^n}{n!}.$$

Explicitly the first few values of the sequence  $B_{3,n}^{sin}$  are given in the table:

n	0	1	2	3	4	5	6	7	8	9	10	11
$B_{3,n}^{sin}$	-1	0	-1/10	0	-11/350	0	-17/1050	0	-563/57750	0	-381/250250	0

Let Sin be the species such that

$$Sin(x) = \begin{cases} (-1)^n & \text{if } |x| = 2n + 1\\ 0 & \text{if } x \text{ is even} \end{cases}$$

Clearly |Sin| = sin,  $B_{2L+1}^{Sin}(x) = 0$  if |x| is odd, and if |x| is even then

$$B_{2L+1}^{Sin}(x) = \bigsqcup_{x_1 \sqcup \ldots \sqcup x_k = x, k \ge 1} (-1)^{\frac{|x|}{2} + kL} (-(2L+1)!)^k \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [2L+1]),$$

where the cardinality of each set  $x_i$  is even. Therefore we obtain that

$$B_{2L+1,2n}^{sin} = |B_{2L+1}^{Sin}[2n]| = (-1)^n 2n! \sum_{2a_1 + \dots + 2a_k = 2n, k \ge 1} \frac{(-1)^{k(L+1)}(2L+1)!^k}{(2a_1 + 2L+1)! \dots (2a_k + 2L+1)!}.$$

Similarly one can consider Bernoulli numbers  $B_{N,n}^{\cos}$  associated with the cosine function for N=2L an even number. For N=2 the Bernoulli numbers  $B_{2,n}^{\cos}$  are such that:

$$\frac{x^2/2!}{\cos(x) - 1} = \sum_{n=0}^{\infty} B_{2,n}^{\cos} \frac{x^n}{n!}.$$

The first few values of  $B_{2,n}^{\cos}$  are shown in the table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B_{2,n}^{cos}$	-1	0	-1/6	0	-1/10	0	-5/42	0	-7/30	0	-15/22	0	-7601/2730	0

Let Cos be the species such that

$$Cos(x) = \begin{cases} (-1)^n & \text{if } |x| = 2n \\ 0 & \text{if } x \text{ is odd} \end{cases}$$

Clearly |Cos| = cos,  $B_{2L}^{Cos}(x) = 0$  if |x| is odd, and if |x| is even then

$$B_{2L}^{Cos}(x) = \bigsqcup_{x_1 \sqcup \ldots \sqcup x_k = x, k \ge 1} (-1)^{\frac{|x|}{2} + kL} (-2L!)^k \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [N]),$$

where the cardinality of each set  $x_i$  is even. Therefore we obtain that

$$B_{2L,2n}^{cos} = |B_{2L}^{Cos}[2n]| = (-1)^n 2n! \sum_{2a_1 + \dots + 2a_k = 2n, k \ge 1} \frac{(-1)^{k(L+1)} 2L!^k}{(2a_1 + 2L)! \dots (2a_k + 2L)!}.$$

# 4 Bernoulli numbers for rational species

In this section we consider Bernoulli numbers associated with formal power series with rational coefficients. Let M be a positive integer and  $\mathbb{Z}^M : \mathbb{B} \longrightarrow gpd$  be the rational species such that for  $x \in \mathbb{B}$  the groupoid  $\mathbb{Z}^M(x)$  is given by

$$Ob(\mathbb{Z}^M(x)) = \left\{ \begin{array}{ll} \{x\} & \text{ if } \ x \neq \emptyset \\ \emptyset & \text{ if } \ x = \emptyset \end{array} \right.$$

and  $\mathbb{Z}^M(x)(x,x) = \mathbb{Z}^M_{|x|}$  for x nonempty; clearly  $|\mathbb{Z}^M| = \sum_{n=1}^{\infty} \frac{1}{n^M} \frac{x^n}{n!}$ . The first Bernoulli numbers for  $|\mathbb{Z}|$  are given in the table:

n	0	1	2	3	4	5	6	7
$B_{1,n}^{\mathbb{Z}}$	1	-1/4	1/72	1/96	61/21600	-1/640	-12491/5080320	-479/580608

For M=2 we get the species  $\mathbb{Z}^2:\mathbb{B}\longrightarrow gpd$  with generating series  $\left|\mathbb{Z}^2\right|=\sum_{n=1}^\infty\frac{1}{n^2}\frac{x^n}{n!}$ . The first Bernoulli numbers for  $\left|\mathbb{Z}^2\right|$  are shown in the table:

ſ	n	0	1	2	3	4	5	6
	$B_{1,n}^{\mathbb{Z}^2}$	1	-1/8	11/432	1/144	-217/324000	-157/64800	-21503/16669800

According to Theorem 6 the generating series of the species  $B_N^{\mathbb{Z}^M}$  sending  $x \in \mathbb{B}$  by

$$B_N^{\mathbb{Z}^M}(x) = \bigsqcup_{x_1 \sqcup ... \sqcup x_k = x, k \ge 1} (-N!)^k \prod_{i=1}^k \mathbb{Z}_{(N)}(x_i \sqcup [N]) \prod_{i=1}^k \overline{\mathbb{Z}}_{|x_i|+N}^M$$

is  $\sum_{n=0}^{\infty} B_{N,n}^{\mathbb{Z}^M} \frac{x^n}{n!}$ . Therefore we obtain that

$$B_{N,n}^{\mathbb{Z}^M} = |B_N^{\mathbb{Z}^M}([n])| = (-1)^n n! \sum_{a_1 + \dots + a_k = n, k \ge 1} \frac{N!^k}{(a_1 + N)!(a_1 + N)^M \dots (a_k + N)!(a_k + N)^M}.$$

Let  $\mathbb{Z}^{(M)}: \mathbb{B} \longrightarrow gpd$  be the rational species sending  $x \in \mathbb{B}$  into  $\mathbb{Z}^{(M)}(x)$  the groupoid given by

$$Ob(\mathbb{Z}^{(M)}(x)) = \begin{cases} \{x\} & \text{if } x \neq \emptyset \\ \emptyset & \text{if } x = \emptyset, \end{cases}$$

and  $\mathbb{Z}^{(M)}(x)(x,x) = \mathbb{Z}_{|x|} \times \mathbb{Z}_{|x|+1} \times \cdots \times \mathbb{Z}_{|x|+M-1}$  for x non-empty. The generating series of  $\mathbb{Z}^{(M)}$  is

$$\left| \mathbb{Z}^{(M)} \right| = \sum_{n=1}^{\infty} \frac{1}{(n)^{(M)}} \frac{x^n}{n!}.$$

The first Bernoulli numbers for  $|\mathbb{Z}^{(3)}|$  and N=3 are shown in the table:

n	0	1	2	3	4	5	6	7
$B_{3,n}^{\mathbb{Z}^3}$	60	-15/12	9/56	3/64	401/31360	127/50176	-9089/33116160	-192233/264929280

## 5 Generalized Bernoulli polynomials

Let  $f \in \mathbb{Q}[[x]]$  and  $N \geq 1$  be such that  $f_N = 1$ . Bernoulli polynomials  $B_{N,n}^f(x)$  are such that

$$\sum_{n=0}^{\infty} B_{N,n}^{f}(x) \frac{y^{n}}{n!} = \frac{f(xy) (y^{N}/N!)}{f(y) - \pi_{N}(f)(y)}.$$

For example Bernoulli polynomials  $B_n(x)$  and  $B_{2,n}(x)$  are given by the identities:

$$\sum_{n=0}^{\infty} B_n(x) \frac{y^n}{n!} = \frac{e^{xy}y}{e^y - 1} \quad \text{and} \quad \sum_{n=0}^{\infty} B_{2,n}(x) \frac{y^n}{n!} = \frac{e^{xy} \left(y^2 / 2!\right)}{e^y - 1 - y}.$$

It is not hard to check that Bernoulli polynomials  $B_{N,n}^f(x)$  satisfy the recursion:

$$\sum_{k=0}^{n-N} \binom{n}{k} f_{n-k} B_{N,k}^f(x) = \binom{n}{N} f_{n-N} x^{n-N} \text{ for } n \ge N,$$

or, equivalently,  $B_{n,0}^f(x) = f_0$  and

$$B_{N,n}^{f}(x) = \frac{\binom{n+N}{N} f_{n}x^{n} - \sum_{k=0}^{n-1} \binom{n+N}{k} f_{n+N-k} B_{N,k}^{f}(x)}{\binom{n+N}{n}}.$$

Next result writes Bernoulli polynomials  $B_{N,n}^f(x)$  in terms of Bernoulli numbers  $B_{N,n}^f$ . **Theorem 7.** Let  $f \in \mathbb{Q}[[x]]$  and  $N \geq 1$  be such that  $f_N = 1$ , then  $B_{N,n}^f(x)$  is given by

$$B_{N,n}^f(x) = \sum_{k=0}^n \binom{n}{k} B_{N,n-k}^f f_k x^k.$$

Proof.

$$\sum_{n=0}^{\infty}B_{N,n}^f(x)\frac{z^n}{n!}=\left(\sum_{n=0}^{\infty}B_{N,n}^f\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}f_nx^n\frac{z^n}{n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^n\left(\begin{array}{c}n\\k\end{array}\right)B_{N,n-k}^ff_kx^k\right)\frac{z^n}{n!}.$$

Consider the species  $S: \mathbb{B} \longrightarrow gpd$  such that S(x) is the groupoid given by  $Ob(S(x)) = \{x\}$  and  $S(x)(x,x) = S_{|x|}$  for  $x \in \mathbb{B}$ . The generating series of S is  $|S| = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{n!}$ , and the corresponding Bernoulli polynomials are given by

$$\begin{split} B_{1,0}^S(x) &= 1, \quad B_{1,1}^S(x) = x - \frac{1}{4}, \quad B_{1,2}^S(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{5}{72}, \\ B_{1,3}^S(x) &= \frac{1}{6}x^3 - \frac{3}{8}x^2 + \frac{5}{24}x - \frac{1}{48}, \\ B_{1,4}^S(x) &= \frac{1}{24}x^4 - \frac{1}{6}x^3 + \frac{5}{24}x^2 - \frac{1}{12}x + \frac{139}{21600}, \\ B_{1,5}^S(x) &= \frac{1}{120}x^5 - \frac{5}{96}x^4 + \frac{25}{216}x^3 - \frac{5}{48}x^2 + \frac{139}{4320}x - \frac{1}{540}. \end{split}$$

Bernoulli numbers for S are shown in the table:

I	n	0	1	2	3	4	5	6	7	8
Ī	$B_{1,n}^S$	-1	-1/4	5/72	1/48	139/21600	-1/540	859/2540160	71/483840	-9769/36288000

## 6 Compositional Bernoulli numbers

Let  $f^{<-1>}$  be the compositional inverse of the formal power series f, i.e.  $f \circ f^{<-1>} = x = f^{<-1>} \circ f$ . Also if k is a positive integer we define inductively  $f^{<1>} = f$  and  $f^{< k+1>} = f^{< k>} \circ f$ .

**Definition 8.** Let  $f = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$  and  $N \geq 1$  be such that  $f_N = 1$ . The compositional Bernoulli numbers  $C_{N,n}^f$  are given by

$$\sum_{n=1}^{\infty} C_{N,n}^f \frac{x^n}{n!} = \left( N! x^{1-N} (f - \pi_N f) \right)^{<-1>} = \left( \sum_{n=1}^{\infty} \frac{N! n! f_{N+n-1}}{(N+n-1)!} \frac{x^n}{n!} \right)^{<-1>}.$$

In order to compute compositional Bernoulli numbers we use the recursion below which is a direct consequence of Definition 8.

**Proposition 9.** For  $n \geq 1$ ,

$$C_{N,n}^f = \sum_{a_1 + a_2 + \dots + a_k = n} \frac{(-N!n!)^k}{k!} \binom{n}{a_1, a_2, \dots, a_k} C_{N,k}^f \prod_{j=1}^k \frac{f_{N+a_j-1}}{(N+a_j-1)!},$$

where  $k \geq 2$ .

Let us provide a combinatorial interpretation for compositional Bernoulli numbers  $C_{N,n}^f$  assuming that a combinatorial interpretation for f is known, i.e. given  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd we construct  $C_N^F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd such that

$$|C_N^F| = (N!x^{1-N}(|F| - \pi_N(|F|))^{<-1>}.$$

First consider the problem of finding a combinatorial interpretation for  $f^{<-1>}$  assuming that a combinatorial interpretation for f is known. Let  $Par : \mathbb{B} \longrightarrow \mathbb{B}$  be the species sending a finite set x into Par(x) the set of partitions of x, i.e. an element  $\pi$  in Par(x) is a family of non-empty subsets of x such that  $\bigcup_{b \in \pi} b = x$  and  $b \cap c = \emptyset$  for  $b, c \in \pi$ . We write  $a \vdash n$  if n is a positive integer and  $a = (a_1, ..., a_l)$  is a sequence of positive integers such that  $|a| = a_1 + ... + a_l = n$ . The integer l(a) is called the length of a. The generating function of Par is given by

$$|Par| = \sum_{n=1}^{\infty} \left( \sum_{a \vdash n} \frac{1}{l(a)!} \binom{n}{a_1, ..., a_l} \right) \frac{x^n}{n!}.$$

For  $d \geq 1$  consider the species  $Par_d^s : \mathbb{B} \longrightarrow \mathbb{B}$  sending  $x \in \mathbb{B}$  into  $Par_d^s(x)$  the set of d-tuples  $\pi = (\pi_1, ..., \pi_d)$  such that:  $\pi_1 \in Par(x)$ ,  $\pi_i \in Par(\pi_{i-1})$  for  $1 \leq i \leq d$ ,  $|\pi_d| \geq 1$ , and  $|\pi_d| \geq 1$ , and  $|\pi_d| \geq 1$  for  $|\pi_d| \leq 1$ .

**Proposition 10.** The generating series of  $Par_d^s$  is given by

$$|Par_d^s| = \sum_{n=1}^{\infty} \left( \sum_{a_1, \dots, a_d} \frac{1}{l(a_d)!} \prod_{i=1}^d \binom{l(a_{i-1})}{a_{i1}, \dots, a_{il(a_i)}} \right) \frac{x^n}{n!},$$

where  $a_1 \vdash l(a_0) := n$ ,  $a_i \vdash l(a_{i-1})$  for  $1 \le i \le d$ ,  $|l(a_d)| \ge s$ , and  $|l(a_d)| \ge s$ .

Let  $gpd_1^{\mathbb{B}}$  be the full subcategory of rational species such that  $F(\emptyset) = \emptyset$  and F([1]) = 1.

**Proposition 11.** If F belongs to  $gpd_1^{\mathbb{B}}$  and  $d \geq 1$ , then the rational species  $F^{< d+1>}$  is given by

$$F^{< d+1>}(x) = \bigsqcup_{\pi \in Par_d^1(x)} F(\pi_d) \times \prod_{i=1}^d \prod_{b \in \pi_i} F(b).$$

*Proof.* For d = 1 the formula above is the well-known result:

$$F^{<2>}(x) = \bigsqcup_{\pi \in Par(x)} F(\pi) \times \prod_{b \in \pi} F(b).$$

The desired formula follows by induction:

$$F^{< d+1>}(x) = F^{< d>} \circ F = \bigsqcup_{\pi_1 \in Par(x)} \left( F^{< d>}(\pi_1) \times \prod_{b \in \pi_1} F(b) \right).$$

Corollary 12. Let F be in  $gpd_1^{\mathbb{B}}$  and  $d \geq 1$ , then we have that

$$|F^{< d+1>}| = \sum_{n=0}^{\infty} \left( \sum_{a_1, \dots, a_d} \frac{|F|_{l(a_d)}}{l(a_d)!} \prod_{i=1}^d \binom{l(a_{i-1})}{a_{i1}, \dots, a_{il(a_i)}} \prod_{j=1}^{l(a_i)} |F|_{a_{ij}} \right) \frac{x^n}{n!},$$

where  $a_1 \vdash l(a_0) := n$  and  $a_i \vdash l(a_{i-1})$  for  $2 \le i \le d$ .

For F in  $\mathbb{Z}_2$ - $gpd^{\mathbb{B}}$  such that  $F(\emptyset) = F([1]) = 0$  consider the species  $(X + F)^{<-1>}$  in  $\mathbb{Z}_2$ - $gpd_1^{\mathbb{B}}$  sending  $x \in \mathbb{B}$  with  $|x| \geq 2$  into

$$(X+F)^{<-1>}(x) = -F(x) \sqcup \bigsqcup_{\pi \in Par_d^2(x), d \ge 1} (-1)^{d+1} F(\pi_d) \times \prod_{i=1}^d \prod_{b \in \pi_i} F(b).$$

The disjoint union above is finite since it is restricted to partitions whose blocks are at least of cardinality two.

**Theorem 13.**  $(X+F)^{<-1>}$  is such that  $|(X+F)^{<-1>}| = (X+|F|)^{<-1>}$ .

*Proof.* The result follows from the identities:

$$(X+F)^{<-1>} = X - F + \sum_{d=1}^{\infty} (-1)^{d+1} F^{< d+1>},$$

$$F^{< d+1>}(x) = \bigsqcup_{\pi \in Par_d^2(x)} F(\pi_d) \times \prod_{i=1}^d \prod_{b \in \pi_i} F(b).$$

Corollary 14. Let  $f = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$  be such that  $f_0 = f_1 = 0$ , then

$$(x+f)^{<-1>} = x + \sum_{n=2}^{\infty} \left( -f_n + \sum_{a_1, \dots, a_d, d \ge 1} (-1)^{d+1} \frac{f_{l(a_d)}}{l(a_d)!} \prod_{i=1}^d \binom{l(a_{i-1})}{a_{i1}, \dots, a_{il(a_i)}} \prod_{j=1}^{l(a_i)} f_{a_{ij}} \right) \frac{x^n}{n!},$$

where  $a_1 \vdash l(a_0) := n$ ,  $a_i \vdash l(a_{i-1})$  for  $2 \le i \le d$ ,  $l(a_i) \ge 2$  and  $a_{ij} \ge 2$ .

**Proposition 15.** Let  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd be a rational species such that F([N]) = 1, then

$$|X + N! \partial^{N-1} (F \times \mathbb{Z}_{(N-1)})| = (N! x^{1-N} (|F| - \pi_N |F|))^{<-1>}$$

From Theorem 13 and Proposition 15 we obtain the promised combinatorial interpretation for the compositional Bernoulli numbers.

**Theorem 16.** For  $F: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd such that F([N]) = 1, let  $C_N^F \in \mathbb{Z}_2$ - $gpd_1^{\mathbb{B}}$  send  $x \in \mathbb{B}$  with  $|x| \geq 2$  into  $C_N^F(x)$ , the disjoint union in  $\mathbb{Z}_2$ -gpd of  $-N!F \times \mathbb{Z}_{(N-1)}(x \sqcup [N-1])$  and

$$\bigsqcup_{\pi \in Par_d^2(x), d \ge 1} (-N!)^{d+1} F \times \mathbb{Z}_{(N-1)}(\pi_d \sqcup [N-1]) \times \prod_{i=1}^d \prod_{b \in \pi_i} F \times \mathbb{Z}_{(N-1)}(b \sqcup [N-1]).$$

Then  $|C_N^F| = \sum_{n=0}^{\infty} C_{N,n}^{|F|} \frac{x^n}{n!}$ .

Recall that we denote by  $E: \mathbb{B} \longrightarrow \mathbb{Z}_2$ -gpd the exponential species.

Corollary 17. Let  $C_N^E : \mathbb{B} \longrightarrow \mathbb{Z}_2$ - $gpd_1^{\mathbb{B}}$  be the species sending  $x \in \mathbb{B}$  with  $|x| \geq 2$  into  $C_N^E(x)$ , the disjoint union of the  $\mathbb{Z}_2$ -graded groupoids  $-N!F \times \mathbb{Z}_{(N-1)}(x \sqcup [N-1])$  and

$$\bigsqcup_{\pi \in Par_d^2(x), d \geq 1} (-N!)^{d+1} \mathbb{Z}_{(N-1)}(\pi_d \sqcup [N-1]) \times \prod_{i=1}^d \prod_{b \in \pi_i} \mathbb{Z}_{(N-1)}(b \sqcup [N-1]).$$

Then  $|C_N^E| = \sum_{n=0}^{\infty} C_{N,n}^{|E|} \frac{x^n}{n!}$ .

Compositional Bernoulli numbers for  $f(x) = e^x$  and N = 1 are such that

$$\sum_{n=1}^{\infty} C_{1,n} \frac{x^n}{n!} = (e^x - 1)^{<-1>} = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \frac{x^n}{n!},$$

thus  $C_{1,n} = (-1)^{n-1}(n-1)!$  for  $n \ge 1$ . For  $f(x) = e^x$  and N = 2 we obtain compositional Bernoulli numbers  $C_{2,n}$  which are such that

$$\sum_{n=1}^{\infty} C_{2,n} \frac{x^n}{n!} = \left(2! \frac{(e^x - 1 - x)}{x}\right)^{<-1>} = \left(\sum_{n=1}^{\infty} \frac{2}{n+1} \frac{x^n}{n!}\right)^{<-1>}.$$

The first compositional Bernoulli numbers  $C_{2,n}$  are shown in the table:

n	0	1	2	3	4	5	6	7	8	9
$C_{2,n}$	0	1	-2/3	5/6	-68/45	193/54	-655/53	19349/540	-57736/405	520343/810

The species  $C_2^E$  in  $\mathbb{Z}_2$ - $gpd_1^{\mathbb{B}}$  sending  $x \in \mathbb{B}$  with  $|x| \geq 2$  into

$$C_2^E(x) = -2\overline{\mathbb{Z}}_{|x|+1} \sqcup \bigsqcup_{\pi \in Par_d^2(x), d \ge 1} (-2)^{d+1} \overline{\mathbb{Z}}_{|\pi_d|+1} \prod_{i=1}^d \prod_{b \in \pi_i} \overline{\mathbb{Z}}_{|b|+1},$$

satisfies  $|C_2^E| = \sum_{n=0}^{\infty} C_{2,n} \frac{x^n}{n!}$ . So we obtain a combinatorial interpretation for the compositional Bernoulli numbers  $C_{2,n}$ :

$$C_{N,n} = |C_2^E([n])| = \frac{-2}{n+1} + \sum_{\pi \in Par_d^2([n]), d \ge 1} \frac{(-2)^{d+1}}{(|\pi_d| + 1) \prod_{i=1}^d \prod_{b \in \pi_i} (|b| + 1)}.$$

Corollary 18.  $C_{N,1} = 1$  and for  $n \ge 2$  we have that

$$C_{N,n} = -\frac{-2}{n+1} + \sum_{a_1,\dots,a_d,d\geq 1} \frac{(-1)^{d+1}}{(l(a_d)+1)!} \prod_{i=1}^d \prod_{j=1}^{l(a_i)} \frac{l(a_{i-1})!}{(a_{i1}+1)!\dots(a_{il(a_i)}+1)!},$$

where  $a_1 \vdash l(a_0) := n$ ,  $a_i \vdash l(a_{i-1})$  for  $1 \le i \le d$ ,  $1 \le i \le d$ ,  $1 \le d$  and  $1 \le d$ .

# 7 Compositional Bernoulli polynomials

Due to the non-commutativity of composition, there are two natural compositional generalizations for Bernoulli polynomials, namely,

$$\left(\sum_{n=1}^{\infty} C_{N,n}^f \frac{y^n}{n!}\right) \circ f(xy) \quad \text{and} \quad f(xy) \circ \left(x, \sum_{n=1}^{\infty} C_{N,n}^f \frac{y^n}{n!}\right).$$

The first generalization is easily studied with the help of the identities:

$$\left(\sum_{n=1}^{\infty} C_{N,n}^{f} \frac{y^{n}}{n!}\right) \circ f(xy) = \left(\sum_{n=1}^{\infty} C_{N,n}^{f} \frac{y^{n}}{n!}\right) \circ \left(\sum_{n=1}^{\infty} (f_{n}x^{n}) \frac{y^{n}}{n!}\right) \\
= \sum_{n=1}^{\infty} \left(\sum_{a \vdash n} \frac{1}{k!} \binom{n}{a_{1}, \dots, a_{k}} C_{N,k}^{f} f_{a_{1}} x^{a_{1}} \dots f_{a_{k}} x^{a_{k}}\right) \frac{y^{n}}{n!} \\
= \sum_{n=1}^{\infty} \left(\sum_{a \vdash n} \frac{1}{k!} \binom{n}{a_{1}, \dots, a_{k}} C_{N,k}^{f} f_{a_{1}} f_{a_{2}} \dots f_{a_{k}} x^{n}\right) \frac{y^{n}}{n!}.$$

The second generalization is less straightforward and is formalized in our next definition.

**Definition 19.** Let  $f = \sum_{n=1}^{\infty} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$  and  $N \geq 1$  be such that  $f_N = 1$ . Compositional Bernoulli polynomials  $C_{N,n}^f(x)$  are such that

$$\sum_{n=1}^{\infty} C_{N,n}^f(x) \frac{y^n}{n!} = f(xy) \circ \left( x, \sum_{n=1}^{\infty} C_{N,n}^f \frac{y^n}{n!} \right).$$

**Theorem 20.** Let  $f \in \mathbb{Q}[[x]]$  and  $N \geq 1$  be such that  $f_N = 1$ , then

$$C_{N,n}^f(x) = \sum_{a \vdash n} \frac{1}{k!} \binom{n}{a_1, \dots, a_k} f_k C_{N,a_1}^f \dots C_{N,a_k}^f x^k.$$

Proof.

$$\sum_{n=1}^{\infty} C_{N,n}^{f}(x) \frac{y^{n}}{n!} = \left( \sum_{n=1}^{\infty} (f_{n}x^{n}) \frac{y^{n}}{n!} \right) \circ \left( x, \sum_{n=1}^{\infty} C_{N,n}^{f} \frac{y^{n}}{n!} \right)$$

$$= \sum_{n=1}^{\infty} \sum_{a \vdash n} \frac{1}{k!} \binom{n}{a_{1}, \dots, a_{k}} f_{k} C_{N,a_{1}}^{f} \dots C_{N,a_{k}}^{f} x^{k} \frac{y^{n}}{n!}.$$

We display compositional Bernoulli polynomials  $C_{N,n}(x) = C_{N,n}^f(x)$  for  $f(x) = e^x$  and N = 1:

$$C_{1,0}(x) = 1, C_{1,1}(x) = x, C_{1,2}(x) = \frac{x^2}{2} - x, C_{1,3}(x) = \frac{x^3}{6} - x^2 - 2x,$$

$$C_{1,4}(x) = \frac{x^4}{24} - \frac{1}{2}x^3 + \frac{5}{2}x^2 - 6x,$$

$$C_{1,5}(x) = \frac{1}{120}x^5 - \frac{1}{6}x^4 + \frac{3}{2}x^3 - 8x^2 + 24x,$$

$$C_{1,6}(x) = \frac{1}{720}x^6 - \frac{1}{24}x^5 - \frac{7}{12}x^4 - \frac{31}{6}x^3 + 32x^2 - 120x,$$

$$C_{1,7}(x) = \frac{x^7}{5040} - \frac{1}{120}x^6 + \frac{1}{6}x^5 - \frac{13}{6}x^4 + 21x^3 - 156x^2 + 720x.$$

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